Elliptic and Krichever formal group laws.

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The formal group law.

A commutative one-dimensional formal group law over a commutative associative ring A is the formal series

$$F(u,v)=u+v+\sum a_{i,j}u^iv^j, \qquad a_{i,j}\in A, \qquad i>0, j>0,$$

satisfying the conditions

$$F(u,v)=F(v,u), \quad F(u,F(v,w))=F(F(u,v),w).$$

The series $f(t)\in A\otimes\mathbb{Q}[[t]]$ uniquely defined by the conditions

$$f(t_1+t_2)=F(f(t_1),f(t_2)), \qquad f(0)=0, \quad f'(0)=1$$

is the exponential of the formal group $F(u,v)$.

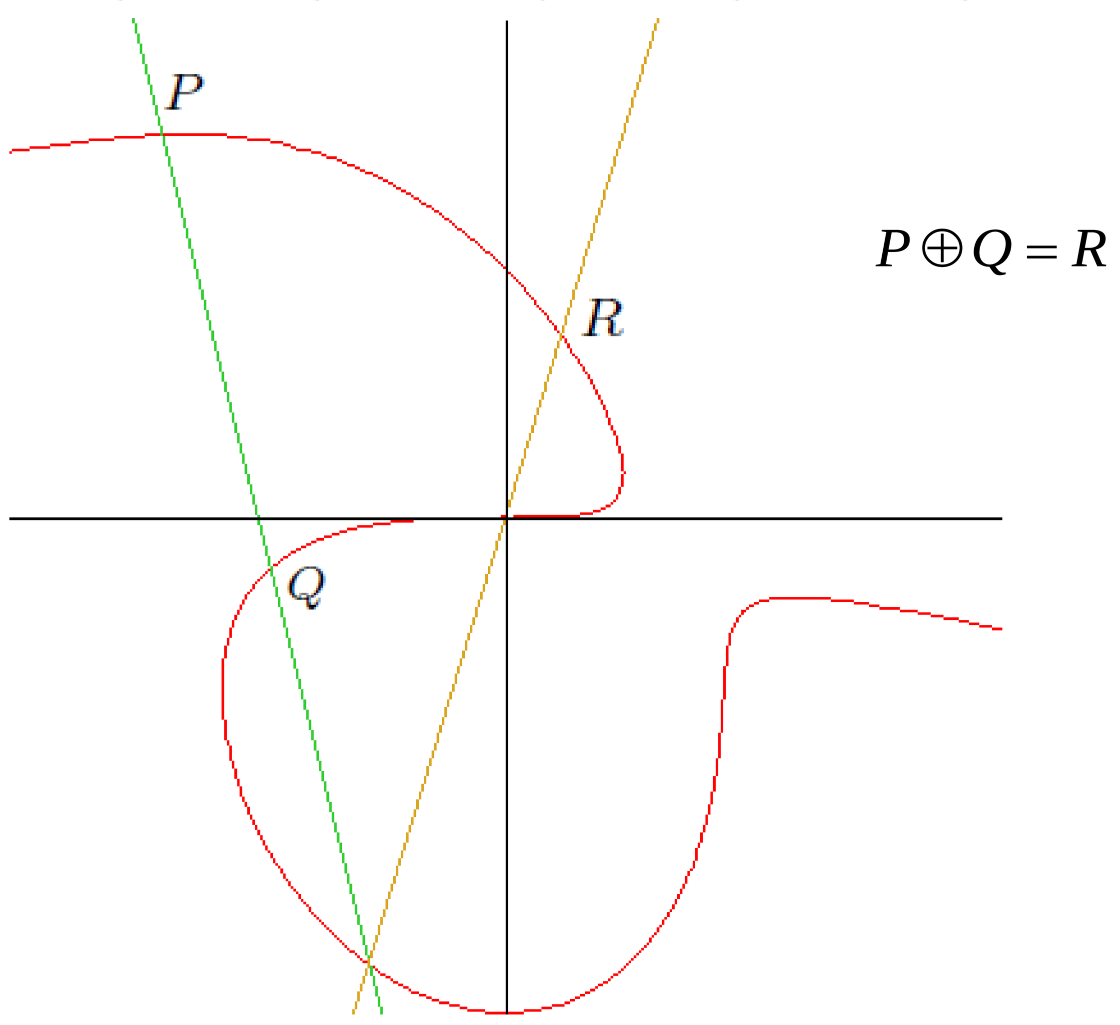
The geometric group structure on the elliptic curve.

The general Weierstrass model of the elliptic curve

$$Y^2Z+\mu_1XYZ+\mu_3YZ^2=X^3+\mu_2X^2Z+\mu_4XZ^2+\mu_6Z^3$$

The arithmetic Tate coordinates $u=-X/Y, \quad s=-Z/Y.$

$$s=u^3+\mu_1us+\mu_2u^2s+\mu_3s^2+\mu_4us^2+\mu_6s^3.$$



The Hirzebruch genera.

Consider the series $f(t)=t+\sum f_kt^{k+1}, \; k\geqslant 1,$ over the ring $A=\sum A_{-2k}$ where $f_k\in A_{-2k}\otimes\mathbb{Q}$.

Set $L_f(\sigma_1,\dots,\sigma_n)=\prod_{i=1}^{n}\frac{t_i}{f(t_i)},$

where σ_k is the k-th elementary symmetric polynomial of $t_1,\dots,t_n$.

The Hirzebruch genus L_f of a stably complex manifold with tangent Chern classes $c_i = c_i\left(\tau(M^{2n})\right)$ is

$$L_f(M^{2n})=\left(L_f(c_1,\dots,c_n),\langle M^{2n}\rangle\right)\in A_{-2n}\otimes\mathbb{Q}.$$

Any genus L_f defines a complex cobordism ring homomorphism $L_f:\Omega_U\to A\otimes\mathbb{Q}$.

The genus L_f is called A-integer if $L_f(M^{2n})\in A$ for any $[M^{2n}]\in\Omega_U$.

The Krichever genus is L_f, where $f(t)=f_{Kr}(t)=\dfrac{\exp(\frac{\mu_1}{2}t)}{\Phi(t)}.$

and $\Phi(t)$ is the Baker-Akhiezer function

$$\Phi(t,\tau)=\frac{\sigma(\tau+t)}{\sigma(t)\sigma(\tau)}e^{-\zeta(\tau)t}.$$

The elliptic formal group law $\mathcal{F}_{El}(u,v)$.

The geometric group structure on the elliptic curve defines the series $\mathcal{F}_{El}(u,v)$ over the ring $E=\mathbb{Z}[\mu_1,\mu_2,\mu_3,\mu_4,\mu_6]$.

Theorem. $$\mathcal{F}_{El}(u,v)=\left(u+v-uv\frac{(\mu_1+\mu_3m)+(\mu_4+2\mu_6m)k}{(1-\mu_3k-\mu_6k^2)}\right)\times$$
$$\times\frac{(1+\mu_2m+\mu_4m^2+\mu_6m^3)}{(1+\mu_2n+\mu_4n^2+\mu_6n^3)(1-\mu_3k-\mu_6k^2)},$$

where

$$m=\frac{s(u)-s(v)}{u-v},\quad k=\frac{us(v)-vs(u)}{u-v},\qquad n=m+uv\frac{(1+\mu_2m+\mu_4m^2+\mu_6m^3)}{(1-\mu_3k-\mu_6k^2)}.$$

Corollary. The formal group law $\mathcal{F}_{El}(u,v)$ gives an $\mathbb{Z}[\mu_1,\mu_2,\mu_3,\mu_4,\mu_6]$-integral Hirzebruch genus.

Differential equations for the exponential.

Set $P(u)=1-\mu_1u-\mu_2u^2$ and $Q(u)=\mu_3+\mu_4u$.

Theorem. The exponential of the elliptic formal group law solves the equation

$$\mu_6\left[f'^3+3P(f)f'^2-4P(f)^3+18P(f)Q(f)f^3+27\mu_6f^6\right]=-Q(f)^2\left[f'^2-P(f)^2+4Q(f)f^3\right].$$

Example. For $\mu_6=0$ we have

$$f'^2=1-2\mu_1f+(\mu_1^2-2\mu_2)f^2+(2\mu_1\mu_2-4\mu_3)f^3+(\mu_2^2-4\mu_4)f^4.$$

Example. For $\mu_6\neq 0$ and $\mu_3=\mu_4=0$ we have

$$f'^3+3P(f)f'^2-4P(f)^3+27\mu_6f^6=0.$$

The Krichever formal group.

Set $\mathcal{B}=\mathbb{Z}[\chi_k:k=1,2,\dots]$. Consider the series of the form

$$\widehat{\mathcal{F}}(u,v)=ub(v)+vb(u)-b'(0)uv+\frac{b(u)\beta(v)-b(v)\beta(u)}{ub(v)-vb(u)}u^2v^2,$$

where $b(u)=1+\sum b_iu^i$, and $\beta(u)=\frac{b'(u)-b'(0)}{2u}=\sum_{k\geqslant0}\beta_{k+2}u^k$.

Here $b_1=\chi_1$, $b_{2i}=\chi_{2i}$ $b_{2i+1}=2\chi_{2i+1}$ $\beta_{2k}=k\chi_{2k}$, $\beta_{2k+1}=(2k+1)\chi_{2k+1}$.

This series defies a formal group $F_{Kr}(u,v)\in\widehat{\mathcal{A}}[[u,v]]$ where $\widehat{\mathcal{A}}=\mathcal{B}/\widehat{\mathcal{J}}$ and $\widehat{\mathcal{J}}$ is the associativity ideal.

Theorem. The exponential of the Krichever formal group is $f_{Kr}(t)$.

Krichever elliptic formal groups.

Theorem.
An elliptic formal group over the ring A with no zero divisors is a Krichever formal group if and only if in A we have:

$$\mu_2\mu_3-\mu_1\mu_4=0, \qquad \mu_3^2+3\mu_6=0, \qquad \mu_3(\mu_1\mu_3+\mu_4)=0.$$

Corollary.
The theorem lists the elliptic Hirzebrugh genera with the rigidity property on S^1-equivariant SU-manifolds.

References.

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