On permutable fuzzy subgroups

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Abstract. A fuzzy subgroup $\kappa$ of a fuzzy group $\gamma$ on a group $G$ is said to be permutable in if $\lambda \circ \kappa = \kappa \circ \lambda$ for every fuzzy subgroup $\lambda$ of $\gamma$. Here $\mu \circ \nu$ stands for the product of two fuzzy groups $\mu$ and $\nu$ on $G$, that is $(\mu \circ \nu)(x) = \lor \{ \mu(y) \land \nu(z) \mid y, z \in G \text{ and } x = yz \}$. In this presentation, largely extending some previous results, we characterize the permutability of fuzzy subgroups in terms of the level subgroups and the support subgroups. We obtain these results emphasizing the role of the characteristic functions of elements of the group. We also show the remarkable fact that the (abstract) subgroups of a group having a fuzzy group whose fuzzy subgroups are permutable, are permutable as well.

Key words: Fuzzy group; fuzzy subgroup; permutable fuzzy subgroup; level subgroup; support subgroup; characteristic function.

Let $G$ be an arbitrary group with a multiplicative binary operation and identity $e$. Recall that a fuzzy subset $\gamma : G \rightarrow [0, 1]$ is said to be a fuzzy group on $G$ (see, for example, [MRB2005, 1.2]), if it satisfies the following conditions:

(FSG 1) $\gamma(xy) \geq \gamma(x) \land \gamma(y)$ for all $x, y \in G$,
(FSG 2) $\gamma(x^{-1}) \geq \gamma(x)$ for every $x \in G$.

We adopt here the usual convention on the operator wedge $\land$ (and on the operator vee $\lor$). If $W$ is a subset of $[0, 1]$, then we denote by $\land W$ the greatest lower bound of $W$, and denote by $\lor W$ the least upper bound of $W$. If $W = \{a, b\}$, then, as usual, instead of $\land W$ we will write $a \land b$, and instead of $\lor W$ we will write $a \lor b$. We assume that the least upper bound of the empty set is 0 and the greatest lower bound is 1.

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There is one important remark we would like to underline at the beginning. We changed one word in this definition: instead of “a fuzzy group of $G$” we say “a fuzzy group on $G$”. Indeed, a fuzzy group is a function defined on a group $G$. Below we will consider relations between fuzzy groups defined on $G$. Let $\gamma, \kappa$ be fuzzy groups on $G$. If $\gamma \subseteq \kappa$, then we say that $\gamma$ is a fuzzy subgroup of $\kappa$ and denotes this by $\gamma \preceq \kappa$. We made this small change in order to avoid possible misunderstanding.

Fuzzy group theory, as other fuzzy algebraic structures, has been introduced soon after fuzzy set theory initial development. Some related results have been collected in the book [MRB2005]. Note that these results not always systematized, many of them are not rigorously formulated, and the methodology and research tools there are at the initial stage. The first natural task here is to describe all fuzzy subgroups of a given fuzzy group $G$. Another main goal of fuzzy group theory is the study of algebraic properties of an arbitrary fuzzy group defined on an abstract group $G$. Here we have the following situation. There are a variety of studies of the structure of the largest fuzzy group $\chi(G, 1)$ on $G$ (note that $\chi(G, 1)$ is the characteristic function of $G$, that is $\text{Im}(\chi(G, 1)) = \{1\}$). In particular, many articles were dedicated to such an important inner property known as to be a normal fuzzy subgroup of $\chi(G, 1)$. Meanwhile, there are significant differences between the case of an arbitrary fuzzy group defined on a group $G$ and the case of the largest fuzzy group $\chi(G, 1)$. If a fuzzy subgroup $\lambda$ of a fuzzy group $\gamma$ defined on a group $G$ possess some given property with respect to $\gamma$, then, in general, it does not always possess the same property with respect to $\chi(G, 1)$. Further, an arbitrary fuzzy group $\gamma$ can be considered as a set of fuzzy points, and this set is a semigroup with identity with respect to the multiplication of the fuzzy sets. However, the largest fuzzy group $\chi(G, 1)$ has many invertible elements (for example, all fuzzy points $\chi(g,1)$, $g \in G$, are invertible), and this makes possible to use essentially the action of the group $G$ on $\chi(G, 1)$. These benefits were very clearly demonstrated in the study of some important properties of normal fuzzy subgroups of $\chi(G, 1)$. At the same time, an arbitrary fuzzy group defined on a group $G$ may have very few invertible elements, and as a consequence, we have very little tangible results on arbitrary fuzzy groups defined on a group $G$. Once again, the example of normal subgroups supports this statement. Our goal is to begin a systematic study of the properties of an arbitrary fuzzy group defined on a group $G$. This is novelty of our work.
One of the key inner properties closely related to the normality is permutability or the property to be a permutable subgroup. Therefore it seems very natural to study it. To define properly the permutable fuzzy subgroup we need to recall the definition of the product of two fuzzy groups.

Define the binary operation “○” on a set of all fuzzy groups by the following rule:

if \( \mu, \nu \) are two fuzzy groups on \( G \), then we put

\[(\mu \circ \nu)(x) = \text{V}_{y, z \in G, yz = x} (\mu(y) \land \nu(z)).\]

We observe that \((\mu \circ \nu)(x) = \text{V}_{y \in G} (\mu(y) \land \nu(y^{-1} x)) = \text{V}_{z \in G} (\mu(xz^{-1}) \land \nu(z)).\)

Remark that in the book [MRB2005], the sign \( \circ \) that has been using for the regular product of mapping was also used for this product. Since we want to avoid any misunderstanding and underline that we are dealing with a completely different operation, we will use the sign \( \circ \).

Let \( \gamma, \kappa \) be fuzzy groups on \( G \). We say that \( \gamma \) and \( \kappa \) permute, if \( \gamma \circ \kappa = \kappa \circ \gamma \). Observe that the product of two fuzzy subgroups in general is not a fuzzy subgroup. More precisely, the product \( \gamma \circ \kappa \) is a fuzzy subgroup if and only if the fuzzy subgroups \( \gamma \) and \( \kappa \) permute (see, for example, [MRB2005, 4.3]).

Let \( \gamma, \kappa \) be a fuzzy group on \( G \) and \( \kappa \preceq \gamma \). We say that \( \kappa \) is permutable in \( \chi \) if \( \lambda \circ \kappa = \kappa \circ \lambda \) for each \( \lambda \preceq \gamma \). The first examples of permutable subgroup are the normal subgroups. Let \( \gamma, \kappa \) be fuzzy groups on \( G \) and suppose that \( \gamma \preceq \kappa \). Recall that \( \gamma \) is a normal fuzzy subgroup of \( \kappa \) if \( \gamma(y x y^{-1}) \geq \gamma(x) \land \kappa(y) \) for every elements \( x, y \in G \) [MRB2005, 1.4]. This fact we will be denoted by \( \gamma \preceq \kappa \).

In abstract group theory, the property to be a permutable subgroup was the main topic of many prolific papers for a long period of time. These papers were devoted to the properties of permutable subgroups in infinite and finite groups. Many of such results can be found in the book by [SCH1994]. For the fuzzy permutable subgroups, the situation is completely different. As for many other properties, the study of this one was initiated only for the largest fuzzy group \( \chi(G, 1) \). In other words, some properties of the permutable fuzzy subgroups of \( \chi(G, 1) \) have been studied. In the case of a finite group \( G \), one can find some initial results of this matter in the papers [AM1991, AA1993, ...]
AA1996, AA1996A], and in the book [MRB2005, 4.3]). Some investigation of permutable fuzzy subgroups of $\chi(G, 1)$ where $G$ is not necessary finite has been initiated in [AT1993, NH2009]. In the current article, we begin the investigation of permutable fuzzy subgroups in arbitrary fuzzy subgroup. Observe, that the transition from the fuzzy group $\chi(G, 1)$ to its fuzzy subgroup significantly complicates the situation (we can observe it even in the case of the property to be normal). Thus, there are many invertible fuzzy points in the group $\chi(G, 1)$. We cannot expect the same in an arbitrary fuzzy subgroup. Nevertheless, in the current article we were able to obtain a significant generalization of the main results of the paper [AT1993].

Recall one more definition. Let $\lambda$ be a fuzzy subset of $X$, $a \in [0, 1]$. Put

$$L_a(\lambda) = \{ x \mid x \in X \text{ and } \lambda(x) \geq a \}.$$  

The subset $L_a(\lambda)$ is called the $a$-level set or a-cut of $\lambda$. Recall that if $\gamma$ is a fuzzy group on $G$, then either $L_a(\lambda)$ is a subgroup of $G$ or $L_a(\lambda) = \emptyset$, in particular, $L_a(\lambda)$ is a subgroup of $G$ for every $a \leq \gamma(e)$. Moreover, using the level sets we can obtain the following characterization of the fuzzy subgroups: $\gamma$ is a fuzzy group on $G$ if and only if $L_a(\lambda)$ is a subgroup of $G$ for each $a \leq \gamma(e)$ (see, for example [MRB2005, Theorem 1.2.6]). Or, $\gamma$ is a fuzzy group on $G$ if and only if every non-empty level of $\lambda$ is a subgroup of $G$.

Our main result is the following

**Theorem A.** Let $G$ be a group and $\gamma$ be a fuzzy group on $G$. A fuzzy subgroup $\kappa$ of $\gamma$ is permutable in $\gamma$ if and only if $L_a(\kappa)$ is a permutable subgroup of $L_a(\gamma)$ for every $a \leq \kappa(e)$.

In the paper [AT1993, Theorem 3.8], a weak version of this result has been proven. Namely, in the mentioned article, a partial case when $\kappa$ is permutable in $\chi(G, 1)$ and satisfies some additional condition (so called the sup property) has been considered.

Here are some corollaries of our theorem.

**Corollary A1.** Let $G$ be a group and $\gamma$ be a fuzzy group on $G$. If a fuzzy subgroup $\kappa$ of $\gamma$ is permutable in $\gamma$ then $\text{Supp}(\kappa)$ is permutable in $\text{Supp}(\gamma)$. 
**Corollary A2.** Let $G$ be a group and $\gamma$ be a fuzzy group on $G$. If a fuzzy subgroup $\kappa$ of $\gamma$ is normal in $\gamma$, then $\kappa$ is a permutable subgroup of $\gamma$.

**Corollary A3.** Let $G$ be a group and $\gamma$ be a fuzzy group on $G$. Suppose that $\lambda, \kappa$ are the permutable fuzzy subgroup of $\gamma$. Then $<\lambda, \kappa>$ is a permutable subgroup of $\gamma$.

**Corollary B.** Let $G$ be a group and $\gamma$ be a fuzzy group on $G$. If every fuzzy subgroup of $\gamma$ is permutable in $\gamma$ then $G$ is a group of one of the following types:

(I) $G = Dr_{\mathbf{p} \in \mathbf{P}(G)} G_p$ where $G_p$ is the Sylow subgroup of $G$ satisfying the following conditions:

(i) if $p \neq 2$, then $G_p = B_p < a_p >$ where $B_p$ is a normal abelian subgroup of exponent $p^k$, and there is a positive integer $t$ such that $t = 1 + p^m, m \leq k \leq m + d$ where $p^d = |G_p/B_p|$ and $a_p^{-1} b a_p = b^t$ for all $b \in B_p$;

(ii) $p = 2$, then either $G_p$ is a Dedekind group or $G_p = B_p <a_p>$ where $B_p$ is a normal abelian subgroup of exponent $p^k$, and there is a positive integer $t$ such that $t = 1 + p^m, 2 \leq m \leq k \leq m+d$ where $p^d = |G_p/B_p|$ and $a_p^{-1} b a_p = b^t$ for all $b \in B_p$.

In both cases, $G_p$ is bounded and nilpotent.

(II) $G$ includes a normal abelian periodic subgroup $T$ such that a factor-group $G/T$ is torsion-free and locally cyclic. Furthermore, every subgroup of $T$ is $G$-invariant.

(III) $G$ is an abelian group.

**Corollary B1.** Let $G$ be a group and $\gamma$ be a fuzzy group on $G$. Then every fuzzy subgroup of $\gamma$ is permutable in $\gamma$ if and only if every subgroup of $\text{Supp}(\gamma)$ is permutable in $\text{Supp}(\gamma)$.

This is not the only result of such nature in fuzzy group theory. For example, every fuzzy subgroup of $\chi(G, 1)$ is normal if and only if every
subgroup of $G$ is normal in $G$ (see [MRB2005, Theorem 4.1.3]). Such kind of results generates the false impression of the easiness of transferring any result from abstract group theory to fuzzy groups defined on $G$. This illusion was supported by a metatheorem of T. Head. T. Head [HT1995] was the first who defined the connections among the sets $\mathcal{P}(X)$ (the power set of $X$), $\mathcal{F}(X)$ (the set of all fuzzy subsets of $X$), and the crisp power set $\mathcal{C}(X)$ of $X$ (the set of characteristic functions of all subsets of $X$) in order to obtain a metatheorem for deriving fuzzy group theorems from their crisp versions. Extending the result of T. Head, A. Weinberger [WA2005] shows that for any set $X$, there is a set $Y$ such that the lattice of fuzzy subsets of $X$ is isomorphic to a sublattice of the classical subsets of $Y$. Moreover, if $X$ is infinite, then it is possible to choose $Y = X$. Employing this result, in the papers [WA1998, WA2005] it was proved that the lattice of fuzzy normal subgroups of $\chi(G, 1)$ is modular (see also the paper [JA2002]), a result that has been originally proven in the papers [AN1995, AT1994]. However, the general situation is much more sophisticated. Among others there are the following important reasons for that.

The isomorphism of lattices of two algebraic structures does not always imply similarity of the properties of these structures. As an example recall the classical Whitman Theorem stating that every lattice is isomorphic to a sublattice of a subgroup lattice of some group.

Abstract groups and fuzzy groups are different algebraic structures. A fuzzy group on a group $G$ as a set of fuzzy points is a semigroup, and as we mentioned above, the set of its invertible elements can be quite small. Therefore, there is no complete correspondence between the concepts that arises in abstract group theory and its fuzzy counterparts. For example, there is a notion of nilpotency in group theory, ring theory, and Lie ring theory. Nevertheless, each of these structures possesses their own characteristics, and not all of the properties pertaining to one of these structures can be transferred to others. We will support this by an example below.

As we mentioned above, in the papers [WA1998, WA2005, JA2002], the authors by employing the metatheorem obtained another proofs yielding that the lattice of normal fuzzy subgroups of $\chi(G, 1)$ is modular. But these proofs (taking into account all the preliminaries and pre-constructions) are not easier than the mentioned original direct proofs of [AN1995, AT1994].

And the last remark: We oconsider mappings of a group $G$ in $[0; 1]$. However, at this stage, we did not employ specific properties of real numbers of $[0; 1]$. The essential fact is that the image of a group is a complete lattice.
As we mentioned above, some results of abstract group theory have corresponding analogs in fuzzy group theory. We will construct the following example showing that this case is not always legitimate. Recall some needed notions from group theory.

Let \( p \) be a prime. For each natural number \( n \geq 1 \) consider a cyclic group \( G_n = \langle a_n \rangle \) of order \( p^n \), and for each \( n \in \mathbb{N} \) denote by \( \theta_n : G_n \rightarrow G_{n+1} \) the monomorphism defined by \( a_n \theta_n = a_{n+1}^p \). In this way we can think of \( G_n \) as a subgroup of \( G_{n+1} \), and hence we can construct the group \( C_p^\infty = \bigcup_{n \in \mathbb{N}} G_n \), which is a union of an ascending chain of cyclic \( p \)-groups of orders \( p, p^2, \ldots \). The obtained above group \( C_p^\infty \) is called a quasicyclic \( p \)-group or a Prüfer groups of type \( p^\infty \).

It is not hard to see, that every proper subgroup of \( C_p^\infty \) is finite. In particular, it satisfies the minimal condition for all subgroups. A group \( G \) is called a Chernikov group, if \( G \) includes a normal subgroup \( D \) of finite index which is a direct product of finitely many Prüfer \( p \)-subgroups. Such groups were named in honour of S.N. Chernikov who made an extensive study of groups with the minimum condition. In particular, S.N. Chernikov proved [ChS1940] that a locally soluble group satisfying the minimal condition for all subgroups is a Chernikov group. Consider now a fuzzy group defined on a quasicyclic \( p \)-group \( G \).

1. Proposition. Let \( G \) be a group and let \(< e > = L_0 \leq L_1 \leq \ldots \leq L_n \leq L_{n+1} \leq \ldots \cup_{n \in \mathbb{N}} L_n = G \) be the countable ascending chain of subgroups. Let \( \{ a_n / n \in \mathbb{N} \} \) be a subset of \([0, 1]\) such that \( a_n \geq a_{n+1} \) for each \( n \in \mathbb{N} \) (it is possible and \( a_n = a_{n+1} \)). Define the function \( \gamma : G \rightarrow [0, 1] \) by the following rule: if \( g \) be an element of \( G \), then let \( k(g) \) be the least number with the property \( g \in L_{k(g)} \). Put \( \gamma(g) = a_{k(g)} \). Then \( \gamma \) is a fuzzy group on \( G \) such that \( L_n \leq L_{\alpha n}(\gamma) \), \( n \in \mathbb{N} \).

Proof. Let \( x, y \) be the arbitrary elements of \( G \). Without loss of generality, we can suppose that \( k(x) \leq k(y) \). Then \( x, y \in L_{k(y)} \). We have \( \gamma(x) = a_{k(x)}, \gamma(y) = a_{k(y)} \). Suppose first that \( k(x) < k(y) \). Then \( y \in L_{k(y) - 1} \), and hence \( y \notin L_{k(x)} \). It follows that \( xy \in L_{k(y)} \), but \( xy \notin L_{k(x)} \). Then \( \gamma(x) = a_{k(x)}, \gamma(y) = a_{k(y)}, \gamma(xy) = a_{k(y)} \), so that \( \gamma(x) \geq \gamma(y) \) and \( \gamma(x) \land \gamma(y) = \gamma(y) \). Hence \( \gamma(xy) = a_{k(y)} = a_{k(y)} \land a_{k(x)} = \gamma(x) \land \gamma(y) \). Suppose now that \( k(x) = k(y) \). In this case, we have the following two possibilities: \( xy \in L_{k(x)} \) and \( xy \notin L_{k(x) - 1} \), or \( xy \in L_m \) for some \( m < k(x) \). In the first case, \( \gamma(xy) = a_{k(x)} = a_{k(x)} \land a_{k(x)} = \gamma(x) \land \gamma(y) \). In the second case, \( \gamma(xy) = a_m \).
\[ a_k(x) = a_k(x) \land a_k(x) = \gamma(x) \land \gamma(y) \]. Thus we can see that \( \gamma \) satisfies the condition (FSG 1). Clearly \( k(x^{-1}) = k(x) \), which implies that \( \gamma(x) = \gamma(x^{-1}) \) for every element \( x \in G \), so that \( \gamma \) satisfies and condition (FSG 2). The inclusion \( L_n \leq L_{an}(\gamma) \), \( n \in N \), follows from the choice of \( \gamma \).

Let now \( G \) be a quasicyclic \( p \)-group, where \( p \) is a prime. Then \( G \) as an ascending chain of cyclic subgroups

\[
< e > \leq < a_1 > \leq \ldots \leq < a_n > \leq < a_{n+1} > \leq \ldots \cup_{n \in N} < a_n > = G,
\]

where \( a_1^p = e, a_{n+1}^p = a_n \) for all \( n \geq 1 \). Using Proposition 1 we will construct now the fuzzy groups \( \gamma_n : G \to [0, 1] \). Let \( 0 < d < 1 \), put \( \gamma_1(x) = d_1 = d \) for all \( x \in G \). Put \( d_2 = d_1/2 \), and define the function \( \gamma_2 \) by the rule: if \( x \in < a_1 > \) then put \( \gamma_2(x) = d_2 \); if \( x \in G \setminus < a_1 > \) then put \( \gamma_2(x) = d_2/2 \). Further, put \( d_3 = d_2/4 \), and define the function \( \gamma_3 \) by the rule: if \( x \in < a_1 > \) then put \( \gamma_3(x) = d_3 \); if \( x \in < a_2 > \setminus < a_1 > \) then put \( \gamma_3(x) = d_3/2 \); if \( x \in G \setminus < a_2 > \) then put \( \gamma_3(x) = d_3/4 \).

Suppose that we already defined the functions \( \gamma_n \) for all \( n < k \). Put \( d_k = d_{k-1}/(2^{k-1}) \), and define the function \( \gamma_k \) by the rule: if \( x \in < a_1 > \) then put \( \gamma_k(x) = d_k \); if \( x \in < a_2 > \setminus < a_1 > \) then put \( \gamma_k(x) = d_k/2 \); if \( x \in < a_3 > \setminus < a_2 > \) then put \( \gamma_k(x) = d_k/4 \); \ldots, if \( x \in < a_k > \setminus < a_k-1 > \) then put \( \gamma_k(x) = d_k/(2^{k-1}) \); if \( x \in G \setminus < a_k > \) then put \( \gamma_k(x) = d_k/2^k \).

Thus we obtain an infinite family \( \{ \gamma_n/n \in N \} \) of fuzzy groups on \( G \). By the construction, \( \gamma_n+1 \preceq \gamma_n \), and \( \gamma_n+1 \) are not equivalent for all \( n \in N \). Therefore we obtain an infinite descending chain

\[ \gamma_1 \succ \gamma_2 \succ \gamma_3 \succ \ldots \succ \gamma_n \succ \ldots \]

of pair wise non-equivalent fuzzy groups defined on the group \( G \). But the group \( G \) has no infinite descending chain of subgroups.

The diagram below illustrates the considered constrictions.
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